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Self-Similar Singularities of the 3D Euler Equations

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Abstract—Self-similar solutions are considered to the incompressible Euler equations in $\mathbb{R}^3$, where the similarity variable is defined as $\xi = \mathbf{x}/(T-t)^\beta \in \mathbb{R}^3$, $\beta \geq 0$. It is shown that the scaling exponent is bounded above: $\beta \leq 1$. Requiring $\|\mathbf{u}\|_{\mathcal{L}^2} < \infty$ and allowing more than one length scale, it is found $\beta \in [2/5, 1]$. This new result on the self-similar singularity is consistent with known analytical results for blow-up conditions.

Whether smooth solutions of the incompressible Euler equations in three space dimensions develop finite-time singularities remains an open question. Some numerical studies and asymptotic models support the possibility (see for example, an earlier paper [1] and [2] with references therein). Rigorous analyses in [3,4] give necessary and sufficient conditions for singularity formation.

A natural assumption to make about the singularity is its self-similar form. This was first advanced by Leray [5] for the 3D incompressible Navier-Stokes equations. But for the Leray system, no such bounded solutions were found [6], while [7] proves that the only solution in $\mathcal{L}^3(\mathbb{R}^3)$ is trivial. However, the result of [7] does not exclude self-similar solutions which locally satisfy the natural energy estimates. There has been continued interest in searching for self-similar Navier-Stokes singularities [8].

In this letter, we consider self-similar transformations of the 3D Euler equation, where the transformation variable is defined by $\xi = \mathbf{x}/(T-t)^\beta$, $\beta \geq 0$. We show that for existence of a self-similar Euler singularity at time $t = T$ with finite energy, there are necessary bounds on the scaling exponent: $\beta \in [2/5, 1]$. Recent works [9–13] indicate self-similar scalings for Euler singularities, it is then expected that our results here may be complementary with numerical solutions for further investigation.

DEFINITION 1. *Let $\mathbf{x} \in \mathbb{R}^3$ be a point in space and $t > 0$ be time. A function f is called self-similar in time if it is in the form*

$$f(\mathbf{x}, t) = \frac{1}{(T-t)^\alpha} F\left(\frac{\mathbf{x}}{(T-t)^\beta}\right), \tag{1}$$

where $\alpha, \beta \geq 0$, $T < \infty$.

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We are interested in self-similar solutions in the above form to the Euler equations. Recall the incompressible Euler equations in $\mathbb{R}^3$,

$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} &= -\nabla p, \\ \nabla \cdot \mathbf{u} &= 0, \end{aligned} \tag{2}$$

where $\mathbf{u}$ denotes the velocity field, p the pressure. The Euler equations in the vorticity formulation are given by

$$\begin{aligned} \frac{\partial \boldsymbol{\omega}}{\partial t} - \nabla \wedge (\mathbf{u} \wedge \boldsymbol{\omega}) &= 0, \\ \nabla \cdot \boldsymbol{\omega} &= 0, \end{aligned} \tag{3}$$

where $\boldsymbol{\omega} = \nabla \wedge \mathbf{u}$. Equations (2) and (3) can be rescaled by introducing the self-similarity.

THEOREM 2. *In accordance with Definition 1, let*

$$\mathbf{u}(\mathbf{x}, t) = \frac{1}{(T-t)^\alpha} \mathbf{U}\left(\frac{\mathbf{x}}{(T-t)^\beta}\right), \qquad p(\mathbf{x}, t) = \frac{1}{(T-t)^{2\alpha}} P\left(\frac{\mathbf{x}}{(T-t)^\beta}\right), \tag{4}$$

where $\mathbf{U}$ *and* P *are functions of*

$$\xi = \frac{\mathbf{x}}{(T-t)^\beta} \in \mathbb{R}^3.$$

Then Euler equation (2) can be transformed into

$$\begin{aligned} \alpha \mathbf{U} + \beta \xi \cdot \nabla \mathbf{U} + \mathbf{U} \cdot \nabla \mathbf{U} &= -\nabla P, \\ \nabla \cdot \mathbf{U} &= 0, \end{aligned} \tag{5}$$

only if

$$\alpha + \beta = 1. \tag{6}$$

PROOF. Substituting (4) into (2), we have

$$\frac{\alpha}{(T-t)^{\alpha+1}} \mathbf{U} + \frac{\beta}{(T-t)^{\alpha+1}} \xi \cdot \nabla \mathbf{U} + \frac{1}{(T-t)^{2\alpha+\beta}} \mathbf{U} \cdot \nabla \mathbf{U} = \frac{-1}{(T-t)^{2\alpha+\beta}} \nabla P.$$

By equating exponents of $(T-t)$ in the above equation, we complete the proof. ∎

REMARK A. For $\alpha < 0$ in (4), one can show that if $\mathbf{U}$ is localized such that $|\mathbf{U}| = O(|\xi|^{-k})$ for $k > 0$ as $|\xi| \to \infty$, then $\mathbf{u} \to 0$ as $t \to T \ \forall \mathbf{x} \neq 0$, which would not seem to be a suitable transformation of the original Euler equations. For this reason, the present paper is restricted to a class of self-similar solutions with $\alpha \geq 0$.

If different scaling exponents are used for pressure p in (4), the theorem is still true.

This theorem gives a constraint on scaling exponents α and β. An immediate consequence of it is the following corollary.

COROLLARY 3. *If* $\mathbf{u}$ *is defined in (4) with* $\mathbf{U}$ *satisfying (5), then*

$$\beta \leq 1.$$

PROOF. From (6),

$$\beta = 1 - \alpha.$$

It is required in Definition 1 that $\alpha \geq 0$ and $\beta \geq 0$, therefore β cannot exceed 1. ∎

Furthermore, by Theorem 2, the vorticity $\boldsymbol{\omega}$ and (3) are transformed into

$$\boldsymbol{\omega} = \frac{1}{(T-t)^{\alpha+\beta}} \nabla \wedge \mathbf{U} = \frac{1}{T-t} \boldsymbol{\Omega}(\xi), \qquad \boldsymbol{\Omega} \equiv \nabla \wedge \mathbf{U}, \tag{7}$$

where $\mathbf{\Omega}$ satisfies the system,

$$\begin{aligned} \mathbf{\Omega} + \beta\xi \cdot \nabla\mathbf{\Omega} &= \nabla \wedge (\mathbf{U} \wedge \mathbf{\Omega}), \\ \nabla \cdot \mathbf{\Omega} &= 0. \end{aligned} \tag{8}$$

REMARK B. If $\beta = 1/2$, (8) is identical to (8.1) in [8] as the inviscid limit of the 3D Navier-Stokes equations. In this case, $\alpha = 1 - \beta = 1/2$.

If a bounded solution $\mathbf{U} \not\equiv 0$ is found by solving (5) or (8), then the Euler system (2) develops a singularity at time $t = T$. Let us state what we mean by a self-similar finite-time singularity of the 3D Euler equations.

DEFINITION 4. *Let* $\mathbf{u}$ *be a smooth solution to (2). We say there is a singularity at time* $t = T$ *if* $\mathbf{u}$ *defined by (4) has finite kinetic energy such that*

$$\int_{\mathbb{R}^3} |\mathbf{u}(\mathbf{x},t)|^2\, d\mathbf{x} = \int_{\mathbb{R}^3} |\mathbf{u}(\mathbf{x},0)|^2\, d\mathbf{x} < \infty, \qquad \text{as } t \to T, \tag{9}$$

but

$$\int_{\mathbb{R}^3} |\boldsymbol{\omega}(\mathbf{x},t)|^p\, d\mathbf{x} = \infty, \qquad \text{as } t \to T \text{ for some } 1 \le p \le \infty, \tag{10}$$

where for $p = \infty$ *it is meant that*

$$\|\boldsymbol{\omega}(\cdot,t)\|_{\mathcal{L}^\infty} = \max_{\mathbf{x}\in\mathbb{R}^3} |\boldsymbol{\omega}(\mathbf{x},t)| = \infty, \qquad \text{as } t \to T.$$

With this definition, we shall find out what set β is in. Note that Corollary 3 establishes $\beta \le 1$, which will be shown to be consistent with the above definition.

LEMMA 5. (FAST DECAY AT ∞). *Assume that in (5),*

$$|\mathbf{U}| = O\left(|\xi|^{-k}\right), \quad |P| = O\left(|\xi|^{-m}\right), \qquad \text{as } \xi \to \infty,$$

where $k > 3/2$, $m > 1/2$. *If the fluid at infinity is at rest, then for* $\mathbf{U} \not\equiv 0$,

$$\beta = \frac{2}{5}.$$

PROOF. Taking the inner-product of (5) and integrating by parts, we have

$$\begin{aligned} &\alpha \int_{\mathbb{R}^3} |\mathbf{U}|^2\, d\xi - \frac{3\beta}{2} \int_{\mathbb{R}^3} |\mathbf{U}|^2\, d\xi \\ &= - \int_{\partial\Omega} \mathbf{U}\cdot\mathbf{n}\; P\, d\xi - \beta \int_{\partial\Omega} \xi \cdot \mathbf{n}\, \frac{|\mathbf{U}|^2}{2}\, d\xi - \int_{\partial\Omega} \mathbf{U}\cdot\mathbf{n} \frac{|\mathbf{U}|^2}{2}\, d\xi, \end{aligned} \tag{11}$$

where $\Omega \subset \mathbb{R}^3$ with $\partial\Omega$ as its surface. Then by the assumptions, the boundary terms on the right-hand side vanish as $\partial\Omega \to \infty$. Therefore, $\mathbf{U} \not\equiv 0$ implies $2\alpha = 3\beta$, which gives $\beta = 2/5$ by Theorem 2. ∎

REMARK C. In [14], $\beta = 2/5$ was obtained by a Hölder seminorm argument, called exact self-similarity. The above proof shows that the exact similarity implies there is one length-scale only, or a flow without outer solutions, which seems highly unlikely (see an argument in [8, p. 13]). Next, we shall deduce from Lemma 5 that if more than one length-scale is allowed, then β need not equal $2/5$.

COROLLARY 6. (SLOW DECAY AT ∞). *Suppose that in Lemm 5 the fluid at infinity is not at rest, and also that for*

$$k \leq \frac{3}{2} \quad \text{and} \quad m \leq \frac{1}{2},$$

the boundary integrations in (11) are finite but nonzero. Then

$$\beta \neq \frac{2}{5}.$$

PROOF. By the assumption, the boundary terms are finite as $\partial\Omega \to \infty$. Denote this constant by C, i.e.,

$$\alpha \int_{\mathbb{R}^3} |\mathbf{U}|^2 \, d\xi - \frac{3\beta}{2} \int_{\mathbb{R}^3} |\mathbf{U}|^2 \, d\xi = C. \tag{12}$$

Since $C \neq 0$, then $2\alpha \neq 3\beta$, that is, $\beta \neq 2/5$ by (6). ∎

The following theorem provides restrictions on β, according to Definition 4.

THEOREM 7. *Assume that*

(i) $\mathbf{U}$ *is a bounded solution of (5) satisfying* $\|\mathbf{U}\|_{\mathcal{L}^2} < \infty$, *and*
(ii) *for each* $\beta > 0$, *there exist at least some* $p > 3\beta$ *such that* $\|\mathbf{\Omega}\|_{\mathcal{L}^p} < \infty$, $1 \leq p \leq \infty$.

Then for a singularity in Definition 4, it is necessary

$$\beta \in \left[\frac{2}{5}, 1\right].$$

PROOF. Norms for self-similar $\mathbf{u}$ and $\boldsymbol{\omega}$ are given by, using (4) and (7),

$$\int_{\mathbb{R}^3} |\mathbf{u}(\mathbf{x}, t)|^2 \, d\mathbf{x} = (T-t)^{3\beta - 2\alpha} \int_{\mathbb{R}^3} |\mathbf{U}(\xi)|^2 \, d\xi, \tag{13}$$

and

$$\int_{\mathbb{R}^3} |\boldsymbol{\omega}(\mathbf{x}, t)|^p \, d\mathbf{x} = (T-t)^{3\beta - p} \int_{\mathbb{R}^3} |\mathbf{\Omega}(\xi)|^p \, d\xi, \qquad 1 \leq p < \infty, \tag{14}$$

in particular by (10) for $p = \infty$,

$$\|\boldsymbol{\omega}(\cdot, t)\|_{\mathcal{L}_\infty} = \frac{\|\mathbf{\Omega}\|_{\mathcal{L}^\infty}}{T-t}, \qquad \|\mathbf{\Omega}\|_{\mathcal{L}^\infty} = \max_{\mathbf{x} \in \mathbb{R}^3} \left| \mathbf{\Omega}\left(\frac{\mathbf{x}}{(T-t)^\beta}\right) \right|. \tag{15}$$

Equation (9) and the above Assumption (i) imply

$$\text{as } t \to T, \qquad \int_{\mathbb{R}^3} |\mathbf{u}(\mathbf{x}, t)|^2 \, d\mathbf{x} < \infty \quad \text{and} \quad \int_{\mathbb{R}^3} |\mathbf{U}(\xi)|^2 \, d\xi < \infty.$$

Hence, from (13) this boundedness is only possible if

$$3\beta \geq 2\alpha.$$

By Theorem 2, this gives a lower bound: $\beta \geq 2/5$. An upper bound is given by Corollary 3: $\beta \leq 1$; we see that $\beta = 1$ implies $\alpha = 0$, so (13) is also finite for the upper bound as $t \to T$. Then Assumption (ii) asserts that, $\forall \beta \in [2/5, 1] \, \exists \, 3\beta < p \leq \infty$ such that $t \to T \Rightarrow \int_{\mathbb{R}^3} |\boldsymbol{\omega}(\mathbf{x}, t)|^p \, d\mathbf{x} = \infty$. The proof is complete. ∎

REMARK D. Assumption (ii) appears quite restrictive, and has been imposed only for definiteness of the discussion. Other regularity conditions on $\mathbf{\Omega}(\xi)$, looser or more stringent, could also be imposed to satisfy Definition 4. Were $\mathbf{\Omega}$ not regularized at all, singularities might occur which are not finite-time ones.

The circulation, or the helicity, can also be a constraint in the above discussion, provided relevant material surface be precisely defined. In a vortex filament model of constant circulation [12], it was found $\beta = 1/2$, consistent with the above argument. Based on the finiteness of kinetic energy, Theorem 7 shows there are relations between α, β, and p for existence of a singularity, from which a few observations may be made.

(a) Apparently $\|\omega(\cdot,t)\|_{\mathcal{L}_\infty}$ can blow up for any value of β, but the requirement of $3\beta \geq 2\alpha$ guarantees that the energy is conserved when the blow-up occurs. It also allows $\|\omega(\cdot,t)\|_{\mathcal{L}_\infty}$ to blow up alone, or simultaneously, with some norms of ω becoming infinite.

(b) If $p = 2$, (14) expresses the enstrophy. Thus, for the enstrophy to become unbounded at the singular time, $\beta < 2/3$ is required.

(c) If $\beta = 1$, then $\alpha = 0$ in (13), so the energy is decreasing in this range. Noticeably, $\alpha = 0$ implies that the velocity $\mathbf{u}$ in (4) does not blow up at $t = T$, the only case of $\mathbf{u}$ remaining bounded at the singular time. This structure might relate to what is called "inner region" of a singularity in [9–11].

(d) Suppose that the direction of vorticity is sufficiently smooth in the sense of ([4, p. 562]). It is stated in [4] that if $\omega \in \mathcal{L}^p_{\text{unif, loc}}$ for some $p > 3$, then

$$\int_0^T R(t)^{-1}\,dt = \infty$$

is necessary for a singularity to arise, where $R(t)$ is a function associated with the radius of curvature of vorticity field lines. Applying their theorem to a self-similar singularity in (14), one would infer that $R \to 0$ as $t \to T$:

$$R(t) \sim (T-t)^{p-3\beta}, \qquad p > 3, \quad \beta \in \left[\frac{2}{5}, 1\right].$$

Since R is so localized, the field lines would have to develop very large curvature in all directions during the singular time.

A curious inquiry here is to what extent does regularity of $\mathbf{\Omega}$ affect structure of a self-similar singularity? In what follows, we describe an example with a stronger assumption than that used in Theorem 7: $\mathbf{\Omega}(\xi)$ is sufficiently well behaved on some subsets of $\mathbb{R}^n$.

EXAMPLE 8. Let $\mathbf{\Omega}(\xi)$ be a smooth solution to (8) such that $\mathbf{\Omega} \in C^\infty$ on some neighborhood of a point ξ_0, where $\xi_0 = \mathbf{x}_0/(T-t_0)^\beta$ by (4). Let $\mathbf{\Omega}$ be represented by a Taylor series about ξ_0. Assume the series converges if $\xi \in \overline{B}(\xi_0;r)$, where $\overline{B}(\xi_0;r)$ is a closed ball of radius $r > 0$. It follows from (7)

$$\omega(\mathbf{x},t) = \frac{1}{T-t}\mathbf{\Omega}(\xi) = \frac{1}{T-t}\sum_{n=0}^{\infty}\frac{\mathbf{\Omega}^{(n)}(\xi_0)}{n!}(\xi-\xi_0)^n.$$

Set $\xi_0 = 0$ arbitrarily. Recall $\xi = \mathbf{x}/(T-t)^\beta$, here $\beta \in [2/5, 1]$ by Theorem 7. Suppose the direction of vorticity is uniform in all directions near ξ_0, then only $|\omega| = |\mathbf{\Omega}|/(T-t)$ needs be considered. We then write the above series as

$$|\omega(\mathbf{x},t)| = \frac{|a_0|}{T-t} + \frac{|a_1|}{(T-t)^{1+\beta}}\,|\mathbf{x}|\frac{|a_2|}{(T-t)^{1+2\beta}}\,|\mathbf{x}|^2 + \cdots + \frac{|a_n|}{(T-t)^{1+n\beta}}\,|\mathbf{x}|^n + \cdots, \qquad n = 0, 1, 2, \cdots,$$

where the expansion coefficients, $|a_n|$, are constants.

Our interest is in the limit of $|\omega(\mathbf{x},t)|$ as $t \to T$. Since $|\mathbf{\Omega}|$ is real-valued and bounded on $\overline{B}$, then $|\mathbf{\Omega}(\overline{B})|$ is a bounded subset of $\mathbb{R}$, so it has a supremum, $|\mathbf{\Omega}(q)| = \sup|\mathbf{\Omega}(\overline{B})|$, $q \in \overline{B}$. By (15), this implies there exist some $\mathbf{x}_q$ such that $|\boldsymbol{\omega}(\mathbf{x}_q,t)| = \|\boldsymbol{\omega}(\cdot,t)\|_{\mathcal{L}_\infty}$, where $0 < |\mathbf{x}_q| < r(T-t)^\beta$. By inspection, for a given $r > 0$, $|\mathbf{x}_q| \to 0$ as $t \to T$. Hence, using the Taylor series, we obtain

$$\lim_{t\to T} |\boldsymbol{\omega}(\mathbf{x}_q,t)| = \infty, \qquad \text{with } \|\boldsymbol{\omega}(\mathbf{x}_q,t)| \sim O(\ |T-t|^{-\gamma_s}\ ), \quad \gamma_s = 1. \tag{16}$$

It is derived in [3] that an Euler singularity is formed iff

$$\int_0^t \|\boldsymbol{\omega}(\cdot,s)\|_{\mathcal{L}^\infty}\, ds = \infty, \qquad \text{as } t \to T.$$

This means that if $\|\boldsymbol{\omega}\|_{\mathcal{L}^\infty}$ is algebraic, then

$$\|\boldsymbol{\omega}(\cdot,t)\|_{\mathcal{L}^\infty} \sim O(\ |T-t|^{-\gamma}), \qquad \text{as } t \to T, \quad \gamma \geqq 1.$$

So we see in (16), the formation of a self-similar singularity is consistent with the [3] criterion. This example shows that if a self-similar Euler singularity occurs in the form (7) with $\boldsymbol{\Omega}$ well behaved and uniform in all directions, then the scaling $\gamma_s \equiv 1$, i.e., $\gamma_s > 1$ cannot arise. Finally, we remark that $\gamma_s = 1$ has been found in numerical computations [8,12].

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